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Transfer of stochastic energy towards high modes and its application to diffeomorphism flows on tori

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Abstract

Dissipativity towards high modes is a classical question in hydrodynamics; for some stochastic model this dissipativity can be exactly computed through the asymptotic of a jump process on the modes; this asymptotical study is the object of this paper.

Keywords: Flow of diffeomorphisms; Stochastic Euler equation

Contents

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0. Introduction

Incompressible fluid dynamic with a vanishing viscosity generates two different one parameter groups of unitary operators acting on two different Hilbert spaces.

At the level of configuration space, fluid motion generates a one parameter subgroup g_t of the group G of volume preserving diffeomorphisms of the torus T^d, $d \geqslant 2$ (the underlying physical space in this paper); by taking inverse image $g_t^* f := f \circ g_t$ we get a one parameter group of unitary operators on the Hilbert space $L^2(T^d)$ of complex-valued square integrable functions on T^d (see [7]).

At the level of the phase space, the fluid speed takes its values in the Hilbert space $\mathcal{G}$ of square integrable vector fields with vanishing divergence; the dynamic generates therefore a $\mathcal{G}$-valued curve; following Arnold's geometrization paradigm, *inertial motion* is defined as the parallel transport for the Levi-Civita connection associated to the Riemannian metric defined by the kinetic energy; then the inertial motion defines a one parameter group of unitary operator on $\mathcal{G}$ (see [6]).

A numerical approach to the construction of these unitary operators will proceed by suitable sequence of finite-dimensional approximations; well chosen approximations will preserve the energy conservation, that is the unitarity: then the key remaining issue will be to prove uniform energy conservation, to avoid the convergence of a sequence of unitary operators towards a physically meaningless operator of norm < 1. From a physical point of view uniform energy conservation will correspond to uniform control of *energy dissipation towards high modes*.

In the case of stochastic dynamic invariant in law by translation on the torus, it has been discovered [6,7], that speed of escape of energy towards high modes can be exactly computed in terms of the escape of a Markov jump processes defined on the dual group Z^d of T^d.

This paper is devoted to estimate the asymptotics of those jump processes in the [7] framework. It is therefore an alternative approach to the theory of flows of diffeomorphisms, theory well established in [1–4] and [8–12] among others; we deal here with the special class of volume preserving diffeomorphisms; the qualitative behaviour for this special class is the same as for the general class as presently established in the scientific literature. The methodologies developed here could be also used to construct the lifting of diffemorphism flows to the phase space, using the transfer energy matrix computed in [6]: we shall not touch here this last point.

1. Energy transfer matrix

We present in this section a short summary of notations and results of [7]. The d-dimensional torus is denoted by T^d. The 1-dimensional torus T^1 is identified with the angle θ defined up to an integral multiple of 2π. We identify T^d to a system of angles $(\theta^1, \dots, \theta^d)$ defined up to an integral multiple of 2π.

A basic tool for our investigations will be Fourier developements. Recall that the dual group of T^d is the group Z^d, the lattice of R^d constituted by points with integral coordinates. Then coupling between T^d and its dual Z^d is given by

$$e_k(\theta) := \exp(ik.\theta), \quad k.\theta = k_1\theta_1 + \cdots + k_d\theta_d; \tag{1.1$_\mathrm{a}$}$$

the collection $\{e_k\}_{k \in Z^d}$ constitutes an orthonormal basis of the Hilbert space $L^2(T^d)$ of *complex-valued functions*, square integrable for the measure $\frac{d\theta^{\otimes}}{(2\pi)^d}$. Given a complex function f defined on T^d, its Fourier transform is defined as $\widehat{f}(k) := (f|e_k)$: then f is real if and only if $\widehat{f}(-k) = \overline{\widehat{f}(k)}$.

We denote by $\tilde{Z}^d$ a subset of Z^d such that each class of the relation on Z^2 defined by $k \simeq k'$ if $k + k' = 0$ has a unique representative in $\tilde{Z}^d$.

We denote by G *the group of volume preserving diffeomorphisms of* T^d, *by* $\mathcal{G}$ *its Lie algebra* which is the space of vector fields with vanishing divergence; then $L^2(T^2; R^d)$ induces on $\mathcal{G}$ a canonical Hilbertian structure.

Define

$$\mathcal{E}_k := \{x \in R^d;\ k.x = 0\}, \quad \text{then } \mathcal{E}_{-k} = \mathcal{E}_k.$$

We obtain the wanted orthonormal basis by picking, for $\forall k \neq 0$, an orthonormal basis $\epsilon_k^1, \dots, \epsilon_k^{d-1}$ of each $\mathcal{E}_k$; we make the convention to take $\epsilon_{-k}^\alpha := \epsilon_k^\alpha$. Then

$$\{A_k^\alpha := \epsilon_k^\alpha \cos k.\theta,\ B_k^\alpha := \epsilon_k^\alpha \sin k.\theta\}_{k \in \tilde{Z}^d,\ \alpha \in [1, d-1]} \tag{1.1$_\mathrm{b}$}$$

constitutes an orthonormal basis of $\mathcal{G}$.

Consider the $\mathcal{G}$-valued Gaussian process

$$y(t) := \sum_{k \in \tilde{Z}^2} \lambda_k \bigl(A_k.x_k^2(t) + B_k.x_k^1(t)\bigr), \quad \lambda_k \text{ being a given sequence}, \ \sum_{k \in \tilde{Z}^d} \lambda_k^2 < \infty, \tag{1.2$_\mathrm{a}$}$$

and $\{(x_k^1, x_k^2)\}$ being a sequence of independent $\mathcal{E}_k \oplus \mathcal{E}_k$-valued Brownian motions.

The *G-valued Brownian motion* is the G-valued process $g_{y,t}$ defined by the following Stratonovitch SDE:

$$dg_{y,t} = \circ\, dy(t)\, g_{y,t}, \qquad g_{y,0} = \text{Identity}. \tag{1.2$_\mathrm{b}$}$$

Let $\mathcal{U}$ be the unitary group of the Hilbert space $L^2(T^d)$. Then the *regular representation* is defined as the map

$$j : G \mapsto \mathcal{U} \text{ by associating to } g \in G \text{ the operator } U_g(f) = f \circ g, \quad \forall f \in L^2; \tag{1.3$_\mathrm{a}$}$$

the representation j induces a morphisms $\tilde{j}$ of Lie algebras; define

$$\mathcal{A}_k = \tilde{j}(A_k), \qquad \mathcal{B}_k = \tilde{j}(B_k), \quad \mathcal{A}_k, \mathcal{B}_k \in \mathrm{su}(\mathcal{G}), \tag{1.3$_\mathrm{b}$}$$

the Lie algebra of antisymmetric operators on $\mathcal{G}$.

The *unitary Brownian* is defined as the operator-valued process solution of the Stratonovitch SDE

$$dU_{y,t} = U_{y,t}\Bigl(\sum_{k \in \tilde{Z}^2} \mathcal{A}_k \circ dy_k^2(t) + \mathcal{B}_k \circ dy_k^1(t)\Bigr), \qquad y_k^* := \lambda_k x_k^*. \tag{1.3$_\mathrm{c}$}$$

Given $z \in L^2(T^d)$, let

$$z_{y,t} := U_{y,t}^*(z); \qquad c_{y,t}(s) = (z_{y,t}|e_s), \qquad \xi_t(s) := E\bigl(|c_{y,t}(s)|^2\bigr), \quad s \in Z^d. \tag{1.4$_\mathrm{a}$}$$

Consider on $l^2(Z^d)$ the symmetric matrix $\mathcal{M}$ associated to the negative quadratic form

$$\big(\mathcal{M}(\xi)|\xi\big) = -\sum_{k\in\tilde{Z}^d}\sum_{s\in Z^d}\lambda_k^2\left(|s|^2-\left(s\Big|\frac{k}{|k|}\right)^2\right)\times\big((\xi_s-\xi_{s+k})^2+(\xi_s-\xi_{s-k})^2\big) \tag*{(1.4)$_\mathrm{b}$}$$

Theorem. *Define ξ_t as $\{\xi_t(s)\}_{s\in Z^d}$. Then ξ_t satisfies the ODE*

$$\frac{d\xi_t}{dt} = \mathcal{M}(\xi_t). \tag*{(1.4)$_\mathrm{c}$}$$

Remark. The hydrodynamical transfert of the energy towards high modes is fully described by $(1.4)_\mathrm{c}$. The object of this paper is a qualitative integration of $(1.4)_\mathrm{c}$.

The diagonal term $\mathcal{M}_l^l$ is obtained by taking either $s=l$ or $s+k=l$ or $s-k=l$. Then

$$\mathcal{M}_l^l = -\sum_{k\in\tilde{Z}^d}\lambda_k^2\left(2\left(|l|^2-\left(l\Big|\frac{k}{|k|}\right)^2\right)+|l+k|^2-\left(l+k\Big|\frac{k}{|k|}\right)^2+|l-k|^2-\left(l-k\Big|\frac{k}{|k|}\right)^2\right),$$

$$\mathcal{M}_l^l = -4\sum_{k\in\tilde{Z}^d}\lambda_k^2\left(|l|^2-\left(l\Big|\frac{k}{|k|}\right)^2\right), \tag*{(1.5)$_\mathrm{a}$}$$

$$\big|\mathcal{M}_l^l\big| \leqslant 4|l|^2\sum_{k\in\tilde{Z}^d}\lambda_k^2. \tag*{(1.5)$_\mathrm{b}$}$$

If we assume that λ_k depends only upon $|k|$:

$$\lambda_k = u\big(|k|\big),\quad \text{then there exists constant } c>0 \text{ such that } \mathcal{M}_l^l \simeq -c|l|^2. \tag*{(1.5)$_\mathrm{c}$}$$

The non-diagonal terms of the quadratic form are:

$$2\sum_{k,l\in\tilde{Z}^d}\lambda_k^2\left(|l|^2-\left(l\Big|\frac{k}{|k|}\right)^2\right)\times\Big(\xi_l\xi_{l+k}+\xi_l\xi_{l-k}\Big).$$

This expression proves their positivity. Furthermore,

$$\mathcal{M}_{k+l}^l = \mathcal{M}_{k-l}^l = 2\lambda_k^2\left(|l|^2-\left(l\Big|\frac{k}{|k|}\right)^2\right) \tag*{(1.5)$_\mathrm{d}$}$$

and

$$\text{the sum of coefficients of each column vanishes.} \tag*{(1.5)$_\mathrm{e}$}$$

Rescale the l column of the matrix $\mathcal{M}$ by dividing each term by $-\mathcal{M}_l^l$. Then we obtain a probability measure carried by the complement of l. Making this construction for all l we define a random walk $X(n)$ on $\tilde{Z}^d$. Then

$$\operatorname{Prob}\{X(n+1)=l+k|X(n)=l\}=\frac{\lambda_k^2(|l|^2-(l|\frac{k}{|k|})^2)}{2\sum_{k\in\tilde{Z}^d}\lambda_k^2(|l|^2-(l|\frac{k}{|k|})^2)},\quad k\in Z^d. \tag{1.6$_\mathrm{a}$}$$

Introduce the angle $\psi_{k,l}$ defined by

$$\left(\frac{l}{|l|}\bigg|\frac{k}{|k|}\right)=\cos\psi_{k,l}.$$

Then

$$\operatorname{Prob}_{X(n)=l}\{X(n+1)=l+k\}=\frac{\lambda_k^2\sin^2\psi_{k,l}}{\sum_{k\in Z^d}\lambda_k^2\sin^2\psi_{k,l}},\quad k\in Z^d. \tag{1.6$_\mathrm{b}$}$$

The *jump process* is defined as

$$\eta(t):=X(\varphi(t)), \tag{1.6$_\mathrm{c}$}$$

where the change of clock $\varphi(t)$ is the integer valued function $\varphi(t)$ defined by

$$A(\varphi(t)):=\sum_{l\leqslant\varphi(t)}\frac{1}{-\mathcal{M}_{X(l)}^{X(l)}}\times\varLambda_l\leqslant t<\sum_{l\leqslant\varphi(t)+1}\frac{1}{-\mathcal{M}_{X(l)}^{X(l)}}\times\varLambda_l=:A^+(\varphi(t)), \tag{1.6$_\mathrm{d}$}$$

and where $\{\varLambda_k\}$ is a sequence of independent exponential times; remark that

$$A^+(\varphi(t))-A(\varphi(t))=\frac{1}{-\mathcal{M}_{X(l)}^{X(l)}}\times\varLambda_l\bigg|_{l=\varphi(t)+1}.$$

The jump process is an autonomous Markovian process, and

$$\text{the infinitesimal generator of the process } \eta \text{ is the matrix } \mathcal{M}. \tag{1.6$_\mathrm{e}$}$$

Let $\varOmega_X$, $\varOmega_\eta$ be the probability spaces of the random walk and of the jump process; let $\varOmega_\varLambda$ be the probability space of an infinite sequence of independent exponential variables. Then $\varOmega_\eta$ is a skew product of $\varOmega_X$ by $\varOmega_\varLambda$ in the following sense: *given a realization of the random walk*, that is a collection of $\{X(n)\}_{n\in N}$, then a realization $\{\varLambda_k\}_{k\in N}$ describes the jump process.

2. Criteria for energy conservation

The jump process is said *conservative* if it cannot go at infinity in a finite time. A consequence of conservativeness is the possibility to numerical approximation of the system by finite-dimensional system.

Define the *sojourn time* of the random walk as

$$S_X(q) := \sum_{n \leqslant q} \frac{1}{-\mathcal{M}_{X(n)}^{X(n)}}. \tag{2.1}$$

We emphasize that the sojourn time is a functional defined on the probability space Ω_X of the random walk.

Theorem. *Assume that* $\inf_l |\mathcal{M}_l^l| := \delta > 0$*, hypothesis which is implied by* $(1.5)_\mathrm{c}$*. Fix a trajectory of the random walk and denote by* E^Λ *(respectively* Prob^Λ*), the expectation (respectively the probability) relatively to the exponential clock. Then*

$$\mathrm{Prob}^\Lambda\big\{S\big(\varphi(t)\big) - t > M\big\} \leqslant \exp(-M\delta), \quad \forall M > 0. \tag{2.2$_\mathrm{a}$}$$

For all $c < 1$ *there exists* $c' > 0$ *such that*

$$\mathrm{Prob}^\Lambda\big\{S\big(\varphi(t) + 1\big) - ct < -M\big\} \leqslant \exp(-c'M\delta), \quad \forall M > 0. \tag{2.2$_\mathrm{b}$}$$

Proof. For $v > 0$ we have

$$\exp(tv) \geqslant E^\Lambda\left\{\exp\left(v \sum_{n \leqslant \varphi(t)} \frac{1}{-\mathcal{M}_{X(n)}^{X(n)}} \Lambda^l\right)\right\} = \prod_{n < \varphi(t)} \frac{1}{1 - \frac{v}{-\mathcal{M}_{X(n)}^{X(n)}}};$$

using the inequality $-\log(1-\beta) \geqslant \beta$, $\beta \in [0, 1[$ we get

$$\exp(tv) \geqslant E^\Lambda\big(\exp\big(v \times S\big(\phi(t)\big)\big)\big), \quad v \in [0, \delta[. \tag{2.2$_\mathrm{c}$}$$

Then $(2.2)_\mathrm{a}$ is obtained by Tchebychev. For $v < 0$ we have

$$\exp(tv) \geqslant E^\Lambda\left\{\exp\left(v \sum_{n \leqslant \varphi(t)+1} \frac{1}{-\mathcal{M}_{X(n)}^{X(n)}} \Lambda^l\right)\right\} = \prod_{n < \varphi(t)+1} \frac{1}{1 - \frac{v}{-\mathcal{M}_{X(n)}^{X(n)}}}.$$

Using that for all $c_1 < 1$ we can find $c_2 > 0$ such that holds true the following inequality: $-\log(1-\beta) \leqslant c_1\beta$, $\beta \in [-c_2, 0]$. We get

$$\exp(tv) \geqslant E^\Lambda\big(\exp\big(c_1 v \times S\big(\phi(t)\big) + 1\big)\big), \quad v \in [-\delta c_2, 0].$$

In particular, taking $v = -\delta c_2$, we get

$$\exp(-\delta c_2 t) \geqslant E^\Lambda\big(\exp\big(-c_1 c_2 \delta \times S\big(\phi(t) + 1\big)\big)\big); \tag{2.2$_\mathrm{d}$}$$

and then $(2.2)_\mathrm{b}$ is obtained by Tchebychev. □

Theorem. *The jump process is conservative if and only if*

$$\mathrm{Prob}_{\Omega_X}\left\{\lim_{q \to \infty} S_X(q) = \infty\right\} = 1. \tag{2.3}$$

Proof. The jump process is not conservative if and only if for some fixed t

$$\mathrm{Prob}_{\Omega_\eta}\big(\varphi(t)=\infty\big)>0.$$

Therefore there exists a subset of positive probability $A\subset\Omega_X$ such that with a positive probability on Ω_A we have $\varphi_X(t)=\infty$. By $(2.2)_\mathrm{a}$ we know that $S(\varphi(t))<\infty$ almost surely. Therefore

$$\lim_{q\to\infty} S_X(q)<\infty,\quad \forall X\in A$$

and

$$\mathrm{Prob}_{\Omega_X}\Big\{\lim_{q\to\infty} S_X(q)<\infty\Big\}>0. \qquad \square$$

Theorem. *Assume* $(1.5)_\mathrm{c}$ *and assume the existence of a second moment, that is*

$$\sum_k \lambda_k^2|k|^2<\infty. \tag{2.4$_\mathrm{a}$}$$

Then the jump process is conservative.

Proof. Recall that $\psi_{k,l}$ has been defined as

$$\left(\frac{l}{|l|}\,\Big|\,\frac{k}{|k|}\right)=\cos\psi_{k,l}.$$

Then we have proved in $(1.6)_\mathrm{d}$ that

$$\mathrm{Prob}_{X(n)=l}\big\{X(n+1)=l+k\big\}=\frac{u^2(|k|)\sin^2\psi_{k,l}}{\sum_{k\in Z^d}u^2(|k|)\sin^2\psi_{k,l}}.$$

Passage in radial coordinate for the random walk

$$\big|X(n+1)\big|^2=\big|X(n)\big|^2+2\big(X(n)\big|X(n+1)-X(n)\big)+\big|X(n+1)-X(n)\big|^2,$$
$$E^{\mathcal{F}_n}\big(X(n+1)-X(n)\big)=0,$$
$$E^{\mathcal{F}_n}\big(\big|X(n+1)\big|^2-\big|X(n)\big|^2\big)=\frac{1}{\sum_{k\in Z^d}u^2(|k|)\sin^2\psi_{k,l}}\sum_{k\in Z^d}|k|^2u^2\big(|k|\big)\sin^2\psi_{k,l}=\gamma,$$

$$\lim_{n\to\infty}E^{\mathcal{F}_n}\big(\big|X(n+1)\big|^2-\big|X(n)\big|^2\big)=\gamma, \tag{2.4$_\mathrm{b}$}$$

where γ is a constant positive independent of $X(n)$. Therefore

$$E\big(\big|X(n)\big|^2\big)\simeq n\gamma \tag{2.4$_\mathrm{c}$}$$

and by the law of large numbers

$$\lim_n \frac{1}{n}\big|X(n)\big|^2 = \gamma$$

almost surely; therefore the series $(2.3)_{\mathrm{d}}$ diverges almost surely. □

Theorem. *Assume* $(1.5)_{\mathrm{c}}$ *and assume that*

$$\gamma = \sup\big\{|k|;\ \lambda_k \neq 0\big\} < \infty; \quad \textit{set } \gamma_1 = \sum \lambda_k^2 |k|^2. \tag{2.5$_{\mathrm{a}}$}$$

Define c_d *by*

$$c_d \times \int_0^\pi \cos^{d-2}\theta\, d\theta = \int_0^\pi \cos^2\theta \cos^{d-2}\theta\, d\theta; \quad \textit{then } c_2 = \frac{1}{2},\ c_d < \frac{1}{2},\ d > 2. \tag{2.5$_{\mathrm{b}}$}$$

Consider the process $z : n \mapsto \log|X(n)|$. *Then*

$$z(n) - \gamma_1(1-2c_d)S_X(n) \quad \textit{is asymptotically a martingale.} \tag{2.5$_{\mathrm{c}}$}$$

Furthermore,

$$\begin{aligned} \frac{1}{B} E\Big(\sup_{n \leqslant N}\big[z(n) - \gamma_1(1-2c_d)S_X(n)\big]^2\Big) &\leqslant E\big(\big[z(N) - \gamma_1(1-2c_d)S_X(N)\big]^2\big) \\ &\leqslant B\gamma_1 E\big(S_X(N)\big), \end{aligned} \tag{2.5$_{\mathrm{d}}$}$$

where E denotes the expectation on the probability space Ω_η of the jump process and where B is a numerical constant.

Proof. All the expectations will be taken on Ω_η. Set

$$\delta_n = \frac{X(n+1) - X(n)}{|X(n)|}, \qquad x_n = \frac{X(n)}{|X(n)|}.$$

Then

$$z(n+1) - z(n) = \log\big(1 + 2(\delta_n|x_n) + |\delta_n|^2\big)$$

as $\delta_n = o(1)$ when $|X(n)| \to \infty$, by using Taylor expansion of log, we get

$$E^{\mathcal{F}_n}\big(z(n+1) - z(n)\big) \simeq E^{\mathcal{F}_n}\big(|\delta_n|^2 - 2(\delta_n|x_n)^2\big). \tag{2.5$_{\mathrm{e}}$}$$

We have

$$E^{\mathcal{F}_n}\big((\delta_n|x_n)^2\big) \simeq c_d E^{\mathcal{F}_n}\big(|\delta_n|^2\big).$$

The Burkholder square function is

$$E^{\mathcal{F}_n}\big(\big[y(n+1)-y(n)\big]^2\big) \simeq 4c_d\gamma_1 \times \frac{1}{|X(n)|^2},$$

where $y(n) = z(n) - \gamma_1(1-2c_d)S_X(n)$. We have

$$\sum_0^N E\big(\big[y(n+1)-y(n)\big]^2\big) \simeq 4c_d\gamma_1 S_X(N).$$

Using Burkholder inequalities [5], we get $(2.5)_{\mathrm{d}}$. $\square$

Corollary. *There exists a constant B' such that*

$$E\big(\big[z(N) - \gamma_1(1-2c_d)S_X(N)\big]^2\big) \leqslant B'\sqrt{E\big(\big[S_X(N)\big]^2\big)}. \tag{2.6}$$

3. Regularity of flow of diffeomorphisms

In order to abbreviate the notations, we use the normalization $\sum_k \lambda_k |k|^2 = 1$. The following lemma transfer the regularity on the diffeomorphism flow to asymptotic estimate of the jump process.

Lemma. *Let $k_1,\dots,k_d$ elements in Z^d such that every element $k \in Z^d$ is linear combination of those elements with integer-valued coefficients. Let ν_0 be the measure on $\tilde{Z}^d$ which associate a Dirac mass at every point k_i. Let ν_t be the transform of the jump process at time t of the measure ν_0. Then we have the expectation of the Sobolev norm H^s satisfied the following equivalence*:

$$E\big(\|g_{x,t}\|^2_{H^s}\big) \simeq \int_{\tilde{Z}^d} |k|^{2s}\nu_t(dk). \tag{3.1}$$

Proof. We abbreviate $a_{2s} = A_{k_s}$, $a_{2s-1} = B_{k_s}$. The map

$$\Phi : T^d \mapsto R^{2d}$$

defined by $\theta \mapsto \{a_k(\theta)\}_{k\in[1,2s]}$ is a diffeomorphism of T^d on its image which is a submanifold of R^{2d}. The regularity of the map g will be controlled by

$$\|g\|^2_{H^q} := \sum_{s=1}^{2d}\sum_{k\in Z^d} \big|\big(g^*a_s\big|e_k\big)\big|^2|k|^{2q} =: \big\|g^*a_s\big\|^2_{H^q}. \qquad \square$$

We need to estimate the moments of ν_t, this program will need firstly the minoration of the sojourn functional of the random walk which is developed in the following lemmas.

Lemma. *Consider a random walk on Z^d such that*

$$E^{\mathcal{F}_n}\big(\big|X(n+1)-X(n)\big|^4\big) \leqslant 1, \qquad E^{\mathcal{F}_n}\big(X(n+1)-X(n)\big) = 0. \tag{3.2$_{\mathrm{a}}$}$$

Set $y(n) = |X(n)|^2$. Let T_R be the first time where $y(*) - y(0)$ is greater or equal to R. Then

$$\operatorname{Prob}\big(T_R < \alpha R\big) \leqslant \frac{4\alpha}{(1-\alpha)^2}\left(1 + \frac{y(0)}{R}\right), \quad \alpha < 1. \tag{3.2$_\mathrm{b}$}$$

Proof. The trivial identity

$$X(n+1) = X(n) + \big(X(n+1) - X(n)\big)$$

reads in radial coordinate as follows:

$$\big|X(n+1)\big|^2 = \big|X(n)\big|^2 + 2\big(X(n)\big|X(n+1) - X(n)\big) + \big|X(n+1) - X(n)\big|^2.$$

As $E^{\mathcal{F}_n}(X(n+1) - X(n)) = 0$ we get

$$E^{\mathcal{F}_n}\big(\big|X(n+1)\big|^2 - \big|X(n)\big|^2\big) = E^{\mathcal{F}_n}\big(\big|X(n+1) - X(n)\big|^2\big) \leqslant 1.$$

Define

$$\phi(n) = \sum_{k<n} E^{\mathcal{F}_k}\big(\big|X(k+1) - X(k)\big|^2\big) \leqslant n,$$
$$M_n := y(n) - \phi(n) - y(0).$$

Then M is a scalar-valued martingale vanishing at 0. Let us compute its quadratic variation

$$E^{\mathcal{F}_n}\big(\big[M_{n+1} - M_n\big]^2\big) = 4E^{\mathcal{F}_n}\big(\big[\big(X(n)\big|X(n+1) - X(n)\big)\big]^2\big)$$

which by Schwarz inequality is dominated by $4y(n)$. By the theory of discrete L^2 martingale we have

$$E\big([M_{T_R}]^2\big) < 4\big(R + y(0)\big)E(T_R).$$

Choose $b \in\,]0, R[$ and define $T_R^b = T_R \wedge b$

$$E\big([M_{T_R^b}]^2\big) < 4\big(R + y(0)\big)b,$$
$$E\big([M_{T_R^b}]^2 1_{T_R<b}\big) \leqslant E\big([M_{T_R^b}]^2\big) < 4\big(R + y(0)\big)b.$$

Then the event $\{T_R < b\}$ we have $M_{T_R^b} > R - b$. Using Tchebychev inequality

$$\operatorname{Prob}(T_R < b) \leqslant 4\frac{b(R + y(0))}{(R-b)^2}$$

and we conclude by taking $b = \alpha R$. $\square$

Define a strictly increasing sequence of stopping times:

$$\dots T_k = \inf_n\big\{\big|X(n) - X(T_{k-1})\big| > 3\big|X(T_{k-1})\big|\big\}\dots. \tag{3.3$_\mathrm{a}$}$$

Set

$$\xi_k = \frac{T_k - T_{k-1}}{|X(T_{k-1})|^2} \tag{3.3$_\mathrm{b}$}$$

then

$$\sum_{n \in [T_{q-1}, T_q[} \frac{1}{|X(n)|^2} \geqslant \frac{1}{16}\xi_q. \tag{3.3$_\mathrm{c}$}$$

Set

$$S^\sharp(n) = \sum_{T_q < n} \xi_q \tag{3.3$_\mathrm{d}$}$$

then

$$\frac{1}{16} S^\sharp(n) < S_X(n). \tag{3.3$_\mathrm{e}$}$$

Lemma. *We have*

$$\operatorname{Prob}\{\xi_k \geqslant 0.05\} \geqslant 0.375. \tag{3.3$_\mathrm{f}$}$$

Proof. We apply $(3.3)_\mathrm{b}$ with $R = 4|X(T_{k-1})|^2 = 4y(0)$. Then $\xi_k = \frac{T_R}{y(0)}$ and

$$\operatorname{Prob}\left(\xi_k < \frac{\alpha}{4}\right) < \frac{3\alpha}{(1-\alpha)^2}.$$

Then take $\alpha = 0.2$. □

Lemma. *There exists k_0 such that*

$$\operatorname{Prob}(\xi_k > 50\lambda) < \exp(-\lambda) \quad \forall \lambda > 1,\ \forall k \geqslant k_0. \tag{3.4$_\mathrm{a}$}$$

Proof. Let $n = T_{k-1} + q$,

$$E(T_k \wedge n - T_{k-1}) < 25\left|X(T_{k-1})\right|^2 \times \operatorname{Prob}(T_k - T_{k-1} > q) + E\left(1_{T_k \leqslant n}\left|X(T_k) - X(T_{k-1})\right|^2\right),$$
$$E(T_k \wedge n - T_{k-1}) < 25\left|X(T_{k-1})\right|^2 + E\left(1_{T_k \leqslant n}\left|X(T_k) - X(T_k - 1)\right|^2\right).$$

Set $\zeta_j := |X(j+1) - X(j)|^2$. Let $F(r) := \operatorname{Prob}\{\zeta_j > r\}$ be the function of repartition of ζ_j. Then

$$E(\zeta_j) = \int_0^\infty F(r)\, dr,$$
$$\zeta_q^* := \sup_{j<q} \zeta_j,$$

$$\mathrm{Prob}\big(\zeta_q^* < r\big) = \big(1 - F(r)\big)^q > \exp\big(-qF(r)\big),$$
$$E\big(\zeta_q^*\big) < \int_0^\infty \big(1 - \exp\big(-qF(r)\big)\big)\,dr.$$

Let R_0 be defined by the condition

$$F(R_0) = \frac{1}{q}.$$

Then

$$E\big(\zeta_q^*\big) < \int_0^\infty \big(1 - \exp\big(-qF(r)\big)\big)\,dr < R_0 + q\int_{R_0}^\infty F(r)\,dR.$$

We have

$$\frac{1}{q} = \mathrm{Prob}(\zeta_j > R_0),$$
$$\frac{R_0}{q} = R_0\,\mathrm{Prob}(\zeta_j > R_0) < E(\zeta_j 1_{\zeta_j > R_0}),$$
$$E\big(\zeta_q^*\big) \leqslant 2qE(\zeta_j 1_{\zeta_j > R_0}) = o(q),$$
$$q \times \mathrm{Prob}(T_k - T_{k-1} > q) \leqslant E^{\mathcal{F}_{T_{k-1}}}(T_k \wedge n - T_{k-1}) < 25\big|X(T_{k-1})\big|^2 + o(q).$$

Taking $q = 50|X(T_{k-1})|^2$ we get the announced estimate for $\lambda = \frac{1}{2}$. We conclude by iterating the estimate through the semi-group property. □

Appendix A. Fourier analysis approach to non-conservativeness

We shall proceed as if the random walk $X(*)$ would be translation invariant, which is not the case; nevertheless for $|X(n)|$ large the random walk is asymptotically translation invariant. Invariance by translation makes available Fourier analysis which leads to a more conceptual approach.

Set

$$\lambda_k^2 = \frac{1}{|k|^{d+\delta}}, \quad \delta \in]0, 2[. \tag{A$_\mathrm{a}$}$$

Then, almost surely

$$\sum_{n=1}^{\infty} \frac{1}{1 + |X(n)|^2} < \infty. \tag{A$_\mathrm{b}$}$$

Consider on R^d the function

$$f(\xi) = c\frac{u(|\xi|)}{|\xi|^{d+\delta}} \tag{$\mathrm{A_c}$}$$

where $|*|$ is the Euclidean distance on R^d, where u is a C^∞ function defined on R^+ vanishing in a neighborhood of zero and equal to 1 on $[1, +\infty[$ and where c is a normalizing constant. As $f \in L^1(R^d)$, its Fourier transform

$$\widehat{f}(\eta) := \int_{R^d} \exp(-2i\pi\xi.\eta) f(\xi)\, d\xi$$

is a continuous function of η. As all derivatives of f are as f itself in $L^1(R^d)$, we have that $\widehat{f}$ multiplied by any polynomials stays bounded. We can therefore apply Poisson summation formula:

$$\psi(\theta) := \sum_{q\in Z^d} \widehat{f}(\theta + q) = \sum_{k\in Z^d} f(k) \exp(ik.\theta); \tag{$\mathrm{A_d}$}$$

we choose the normalization constant c such that $\psi(0) = 1$.

The law of $X(n)$ generates a Fourier series having for sum $[\psi(\theta)]^n$. The function $-|\xi|^2$ is the Fourier transform in the sense of distributions of the Laplacian; therefore the function $(|\xi|^2 + 1)^{-2}$ is the Fourier transform

$$g(\eta) := \int_0^{+\infty} \exp\left(-\frac{1}{2t}|\eta|^2 - t\right) \frac{1}{[2\pi t]^{\frac{d}{2}}}\, dt. \tag{$\mathrm{A_e}$}$$

Then outside $\eta = 0$ we have that g is a C^∞ function decreasing rapidly at infinity; nearby $\eta = 0$, g has the same behaviour as the Newtonian potential kernel $c_d|\eta|^{2-d}$, $d > 2$, or $\log|\xi|$, $d = 2$.

According the rapid decreasing of g we can apply again the Poisson summation formula defining

$$G(\theta) := \sum_{q\in Z^d} g(\theta + q).$$

Then

$$G(\theta) = \sum_k \frac{1}{1 + |k|^2} \exp(ik.\theta),$$

$$\int_{T^d} \psi^n(\theta) G(\theta)\, d\theta = E\left(\frac{1}{1 + |X(n)|^2}\right),$$

and by summing

$$E\left(\sum_{n=0}^{\infty}\frac{1}{1+|X(n)|^2}\right)=\int_{T^d}\frac{1}{1-\psi}G\,d\theta. \tag{$\mathrm{A_f}$}$$

As

$$\widehat{f}=v\bigl(|\eta|\bigr),\qquad \bigl|v(\eta)-v(0)\bigr|>c|\eta|^{\delta},$$

we get the convergence of the right-hand side of $(\mathrm{A_f})$.

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